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THE ARTIN-REES PROPERTY AND HOMOLOGY

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ABSTRACT

The Artin-Rees property for a finitely generated nilpotent group G is used to prove that $H_*(G,M) \approx H_*(G,\hat{M})$ for any finitely generated G-module M, where $\hat{M}$ is the completion of M with respect to the augmentation ideal of $\mathbf{Z}[G]$. Applications to topology are given.

Nouazé and Gabriel ([15], 2.7 and 2.8) have shown that the classical Artin-Rees lemma for commutative noetherian rings admits a generalization which applies, for example, to the group ring of a finitely generated nilpotent group. The purpose of the present paper is to give some applications to homological algebra and topology of this generalized Artin-Ress lemma. These applications concern maps (of modules or spaces) which induce homology isomorphisms.

In Section 1 we describe the Artin-Rees property and give the proof that it is satisfied by finitely generated nilpotent groups.

Section 2 contains the applications to homological algebra. The main result, which includes a theorem of Dwyer's [8] as a special case, is the following (2.2, Th. 3): *If G is a finitely generated nilpotent group and M is a finitely generated G-module, then $H_*(G,M) \approx H_*(G,\hat{M})$, where $\hat{M}$ is the completion of M with respect to the augmentation ideal of* $\mathbf{Z}(G]$. This result is used in 2.3 to show that for a large class of groups, including all finite groups and all finitely generated nilpotent groups, $\hat{M}$ is equal to the $H\mathbf{Z}$-localization of M in the sense of Bousfield [2], for any finitely generated G-module M.

In Section 3 we illustrate how the results of Sections 1 and 2 can be used in topology by proving (a) a vanishing theorem for certain homology groups associated to a prenilpotent space (3.1, Th. 5) and (b) a theorem concerning the homotopy groups of a (higher dimensional) knot complement (3.2, Th. 6).

Finally, an appendix contains a result needed in 3.1 concerning the homology (mod $\mathscr{C}$) of a regular covering space, where $\mathscr{C}$ is a Serre class of abelian groups; as an immediate consequence, we obtain a generalization to nilpotent spaces of Serre's mod $\mathscr{C}$ Hurewicz theorem for simply connected spaces.

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1. The Artin-Rees property

Let R be a left noetherian ring and I a two-sided ideal. If M is a (left) R-module, *the I-adic topology* on M is the unique topology which is compatible with the group structure and in which $\{I^n M\}_{n\geq 0}$ is a fundamental system of neighborhoods of 0. (Equivalently, a neighborhood base at 0 is formed by the submodules M' of M such that M/M' is *I-nilpotent*, i.e., is annihilated by a power of I.) We will say that I has the (left) *Artin-Rees property* if for every finitely generated (left) R-module M and every submodule N, the I-adic topology on N coincides with the restriction to N of the I-adic topology on M.

The following reformulation of the definition is essentially due to Gabriel (cf. [10], V, §5, Prop. 9):

PROPOSITION 1. *The following conditions on I are equivalent*:

(i) *I has the Artin-Rees property.*

(ii) *If M is a finitely generated R-module which contains an essential I-nilpotent submodule, then M is I-nilpotent.*[†]

(iii) *If M is a finitely generated R-module which contains an essential submodule N such that $IN = 0$, then M is I-nilpotent.*

(i) $\Leftrightarrow$ (ii): The proof is identical with Gabriel's proof (*loc. cit.*), so we omit it. [Take C, in the notation of [10], to be the category of I-nilpotent modules.]

(ii) $\Rightarrow$ (iii): Trivial.

(iii) $\Rightarrow$ (ii): Let M be as in (ii) and let $N = \{x \in M : Ix = 0\}$. Then it is easy to verify that N is an essential submodule of M, so (ii) follows from (iii).

We now specialize to the case where R is the integral group ring $\mathbf{Z}[G]$ of a group G, and I is the augmentation ideal. We will say that a G-module is *nilpotent* if it is I-nilpotent, and we will say that G has the *Artin-Rees property* if $\mathbf{Z}[G]$ is noetherian and I has the Artin-Rees property. (Note that there is no need to distinguish here between the left and right Artin-Rees properties, since $\mathbf{Z}[G]$ has an anti-automorphism which takes I onto itself.)

The following theorem is a special case of a result due to Nouazé and Gabriel ([15], 2.7 and 2.8); we will give the proof for the convenience of the reader.

[†] Recall that N is said to be an *essential* submodule of M if every non-zero submodule of M intersects N non-trivially.

THEOREM 1. *If G is finitely generated and nilpotent, then G has the Artin-Rees property.*

The proof is based on the well-known fact (cf. [12], proof of Theorem 10.2.4) that G has a central series

$$G = G_1 \supset G_2 \supset \cdots \supset G_n = \{1\}$$

such that each quotient G_i/G_{i+1} is cyclic. In particular, it follows easily that $\mathbf{Z}[G]$ is noetherian (cf. [16], p. 136). Theorem 1 now follows, by induction on the minimal length n of such a series, from:

PROPOSITION 2. *Let G be a group such that $\mathbf{Z}[G]$ is noetherian. If G has a central cyclic subgroup C such that G/C has the Artin-Rees property, then G has the Artin-Rees property.*

We will verify condition (iii) of Proposition 1. Thus we must show that if M is a finitely generated G-module which contains an essential submodule N on which G acts trivially, then M is nilpotent. We will do this by showing (a) that M is nilpotent as a C-module and (b) that M^C, the set of elements of M fixed by C, is nilpotent as a G/C-module (and hence as a G-module). Assuming for the moment that (a) and (b) have been established, we complete the proof as follows. Let $r = 1 - t$, were t is a generator of C. Then r is a central element of $\mathbf{Z}[G]$ and multiplication by r is a G-module endomorphism of M whose kernel is M^C. Using the exact sequences

$$0 \to M^C \to \ker r^n \xrightarrow{r} \ker r^{n-1},$$

we conclude from (b) (by induction on n) that $\ker r^n$ is a nilpotent G-module for each $n \geqq 1$. Since $\ker r^n = M$ for large n by (a), M is indeed nilpotent.

It remains to prove (a) and (b). For (b) we need only note that M^C is a finitely generated G/C-module which contains N as an essential submodule, hence M^C is nilpotent by the assumption on G/C. To prove (a) we consider the ascending chain $\{\ker r^n\}_{n \geq 1}$ of submodules of M. This chain must stabilize since M is finitely generated, and it follows easily that $\ker r^n \cap \operatorname{im} r^n = 0$ for large n. Since $N \subset \ker r^n$ and N is essential, we conclude that $\operatorname{im} r^n = 0$, as required.

REMARK. If G is only assumed to be polycyclic instead of finitely generated nilpotent, then G need not have the Artin-Rees property. For example, let k be the field $\mathbf{Z}/p\mathbf{Z}$, where p is an odd prime, let M be a two-dimensional vector space over k, let M' be a one-dimensional subspace, and let G be the group of automorphisms of M which act as the identity on M'. (Thus G is the matrix group $\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)$.) Then one verifies easily that M' is an essential G-submodule of M, but that M is not a nilpotent G-module (in fact, $IM = M$), so the Artin-Rees property fails. (Note that G is polycyclic, being the semi-direct product of the sdditive and multiplicative groups of k, both of which are cyclic.)

2. Applications to homological algebra

2.1. *Tor and Completion*

Let R be a ring and I a two-sided ideal. If M is a left R-module, we denote by $\hat{M}$ the completion of M with respect to the I-adic topology:

$$\hat{M} = \varprojlim M/I^n M.$$

We denote by α the canonical map $M \to \hat{M}$. In case $M = R$, the completion $\hat{R}$ is a ring and $\alpha: R \to \hat{R}$ is a ring homomorphism, by means of which we regard $\hat{R}$ as an R-bimodule.

For any (left) R-module M we denote by $\bar{M}$ the $\hat{R}$-module obtained from M by extension of scalars:

$$\bar{M} = \hat{R} \otimes_R M.$$

There is a canonical map $\beta: M \to \bar{M}$ and, since $\hat{M}$ has an obvious $\hat{R}$-module structure, there is a unique $\hat{R}$-module homorphism $\gamma; \bar{M} \to \hat{M}$ such that $\gamma\beta = \alpha$:

$$\begin{array}{ccc} & \beta & \\ M & \rightarrow & \bar{M} \\ \alpha \searrow & & \swarrow \gamma \\ & \hat{M} & \end{array}$$

Proposition 3. *Assume that R is left noetherian and that I satisfies the left Artin-Rees property.*

(i) *The functor $M \mapsto \hat{M}$ is exact on the category of finitely generated left R-modules.*

(ii) *If M finitely generated then $\gamma; \bar{M} \to \hat{M}$ is an isomorphism.*

(iii) *$\hat{R}$ is flat as a right R-module.*

(iv) *Let M be a finitely generated left R-module and let $I^\infty M$ be the kernel of* $\alpha: M \to \hat{M}$, i.e. $I^\infty M = \bigcap_{n \geqq 1} I^n M$. *Then $I \cdot I^\infty M = I^\infty M$, and $I^\infty M$ is the largest submodule of M with this property.*

These consequences of Artin-Rees property are proved exactly as in the commutative case. See, for example, [1], Props. 10.12, 10.13 and 10.14, and the proof of Prop. 10.17.

NOTE. Proposition 3 has an obvious analogue for a right noetherian ring and an ideal with the right Artin-Rees property. This analogue will be referred to as Proposition 3_r.

We can now prove the main result of this section:

THEOREM 2. *Assume that R is left and right noetherian and that I satisfies the left and right Artin-Rees properties. For any left R-module M, β induces an isomorphism*

$$\mathrm{Tor}_*^R(R/I, M) \xrightarrow{\approx} \mathrm{Tor}_*^R(R/I, \bar{M});$$

if M is finitely generated, then α induces an isomorphism

$$\mathrm{Tor}_*^R(R/I, M) \xrightarrow{\approx} \mathrm{Tor}_*^R(R/I, \hat{M}).$$

In view of Proposition 3 (ii), it suffices to prove the assertion about β. Since $\hat{R}$ is a flat right R-module (Prop. 3(iii)), the functor $M \mapsto \bar{M}$ is exact, and hence the functors $\mathrm{Tor}_i^R(R/I, \bar{M})$, as functors of the variable M, form a connected exact sequence of functors, in the sense of [4], Chap. V, §4. It therefore suffices to show (*loc. cit.*, Prop. 4.4) (a)) that β induces an isomorphism $R/I \otimes_R M \xrightarrow{\approx} R/I \otimes_R \bar{M}$, and (b) that $\mathrm{Tor}_i(R/I, \bar{M}) = 0$ if $i > 0$ and M is free.

To prove (a), consider the commutative square

$$\begin{array}{ccc} R/I \otimes_R M & \xrightarrow{\delta \otimes_R M} & (R/I \otimes_R \hat{R}) \otimes_R M \\ {\scriptstyle R/I \otimes_R \beta} \downarrow & & \wr\wr \\ R/I \otimes_R \bar{M} & = & R/I \otimes_R (\hat{R} \otimes_R M), \end{array}$$

where $\delta: R/I \to R/I \otimes_R \hat{R}$ is the canonical map, $x \mapsto x \otimes 1$. Now Prop. 3_r(ii) implies that δ can be identified with the canonical map of R/I to its I-adic completion as a right R-module; but R/I is complete, so δ is an isomorphism and (a) follows at once. To prove (b), it suffices to consider the case $M = R$, in which case the result follows from the flatness of $\hat{R}$ as a left R-module (Prop. 3_r(iii)).

COROLLARY 1. *The following are equivalent for a map $f: M \to N$ of finitely generated R-modules:*

(i) *The map $f_i: \mathrm{Tor}_i^R(R/I, M) \to \mathrm{Tor}_i^R(R/I, N)$ induced by f is an isomorphism for all $i \geqq 0$.*

(ii) *f_0 is an isomorphism and f_1 is an epimorphism.*

(iii) *f induces an isomorphism $\hat{f}: \hat{M} \to \hat{N}$.*

In fact, (i) $\Rightarrow$ (ii) trivially; (ii) $\Rightarrow$ (iii) by the argument of [5], proof of Prop. 5.2 (which does not require the Artin-Rees property or finiteness of M and N); and (iii) $\Rightarrow$ (i) by Theorem 2.

In case $N = 0$, the implication (ii) $\Rightarrow$ (i) of Corollary 1 yields:

COROLLARY 2. *If M is a finitely generated left R-module such that* $\mathrm{Tor}_0^R(R/I, M) = 0$, *then* $\mathrm{Tor}_i^R(R/I, M) = 0$ *for all* i.

REMARK. The isomorphisms

$$R/I \otimes_R M \xrightarrow{\approx} R/I \otimes_R \hat{M}$$

and

$$R/I \otimes_R M \xrightarrow{\approx} R/I \otimes_R \bar{M}$$

of Theorem 2 can be proved under much weaker hypotheses than those of the theorem, namely, we need only assume that I is finitely presented as a right R-module (no hypotheses on M). One proves the first isomorphism by computing $R/I \otimes_R \hat{M}$ by means of the short exact sequence

$$0 \to \varprojlim{}^{(1)} \mathrm{Tor}_1^R(N, M_i) \to N \otimes_R \varprojlim M_i \to \varprojlim N \otimes_R M_i \to 0$$

(cf. [17], Th. 2), valid for any tower of R-modules $\{M_i\}$ such that $\varprojlim^{(1)} M_i = 0$ and for any right R-module N such that there exists an exact sequence $F_2 \to F_1 \to F_0 \to N \to 0$ with F_1 finitely generated and free. The second isomorphism can be deduced from the first, applied with $M = R$.

2.2. *The homology of a finitely generated nilpotent group*

Let G be a group with the Artin-Rees property, e.g., a finitely generated nilpotent group (Section 1, Th. 1). Then all of the results of 2.1 apply with $R = \mathbf{Z}[G]$ and I equal to the augmentation ideal. In particular, for any G-module M we have G-modules $\hat{M}$ and $\bar{M}$ and maps $\alpha: M \to \hat{M}$ and $\beta: M \to \bar{M}$, and Theorem 2 and its corollaries yield:

THEOREM 3. *For any G-module M, β induces an isomorphism $H_*(G, M) \xrightarrow{\approx} H_*(G, \bar{M})$; if M is finitely generated, then α induces an isomorphism $H_*(G, M) \xrightarrow{\approx} H_*(G, \hat{M})$.*

COROLLARY 1. *The following are equivalent for a map $f: M \to N$ of finitely generated G-modules:*

(i) *The map $f_i: H_i(G, M) \to H_i(G, N)$ induced by f is an isomorphism for all i.*

(ii) *f_0 is an isomorphism and f_1 is an epimorphism.*

(iii) *f induces an isomorphism $\hat{f}: \hat{M} \to \hat{N}$.*

COROLLARY 2. (Dwyer [8]). *If M is a finitely generated G-module such that $H_0(G,M)=0$, then $H_i(G,M)=0$ for all i.*

2.3. *$H\mathbf{Z}$-localization for modules over prenilpotent groups*

Let G be a group. We recall some terminology from [2]. A map $f: M \to N$ of G-modules is called an *$H\mathbf{Z}$-map* if the induced map $f_i: H_i(G,M) \to H_i(G,N)$ is an isomorphism for $i=0$ and an epimorphism for $i=1$. A G-module M is said to be *$H\mathbf{Z}$-local* if every $H\mathbf{Z}$-map $f: N_1 \to N_2$ induces an isomorphism $\mathrm{Hom}(N_2,M) \xrightarrow{\approx} \mathrm{Hom}(N_1,M)$. Finally, an *$H\mathbf{Z}$-localization* of a G-module M is an $H\mathbf{Z}$-map $f: M \to M'$ with M' $H\mathbf{Z}$-local. It is easy to see any two $H\mathbf{Z}$-localizations of M are canonically isomorphic. Moreover, it is proved in [2] that every G-module M admits an $H\mathbf{Z}$-localization, but we will not need to use this fact.

We call a group G *prenilpotent* if the lower central series $\{\Gamma_i G\}_{i\geq 1}$ stabilizes, i.e., if $\Gamma_i G = \Gamma_{i+1} G$ for large i. (Here $\Gamma_1 G = G$ and $\Gamma_{i+1} G = (G, \Gamma_i G)$, cf. [12], Chap. 10.) For example, every finite group is prenilpotent. The purpose of this section is to prove:

THEOREM 4. *If G is a finitely generated prenilpotent group and M is a finitely generated G-module, then the canonical map $\alpha: M \to \hat{M}$ is the $H\mathbf{Z}$-localization of M.*

(As in 2.2, $\hat{M}$ is the I-adic completion of M, where I is the augmentation ideal of $\mathbf{Z}[G]$.)

We will need the following three lemmas:

LEMMA 1. *For any group G and any G-module M, the I-adic completion $\hat{M}$ is $H\mathbf{Z}$-local.*

In fact, it is immediate from the definition that an inverse limit of local modules is local, so it suffices to show that any nilpotent module M is local. Assume, then that $I^n M = 0$ and let $f: N_1 \to N_2$ be an $H\mathbf{Z}$-map. Then f induces an isomorphism $N_1/I^n{}_1 \xrightarrow{\approx} N_2/I^n N_2$ ([5], Prop. 5.2), and the lemma follows from the diagram

$$\begin{array}{ccc} \mathrm{Hom}(N_2, M) & \longrightarrow & \mathrm{Hom}(N_1, M) \\ \approx \downarrow & & \downarrow \approx \\ \mathrm{Hom}(N_2/I^n N_2, M) & \longrightarrow & \mathrm{Hom}(N_1/I^n N_1, M). \end{array}$$

LEMMA 2. *Let G be a prenilpotent group and let $\Gamma = \Gamma_i G$ for large i. If M is a G-module, then Γ acts trivially on $M/I^n M$ for all n.*

This follows from the elementary fact that if $\gamma \in \Gamma_n G$ then $\gamma - 1 \in I^n$. (See, for example, [14], Chap. I, §3, Th. 3.2; alternatively, prove this fact by induction on n, using the first lemma on p. 138 of [16].)

LEMMA 3. *If G is finitely generated and prenilpotent, then the abelianization Γ_{ab} of Γ is a finitely generated G/Γ-module, where Γ is as in Lemma 2.*

(The action of G/Γ on Γ_{ab} is induced by the conjugation action of G on Γ.)

In fact, if S is a finite set of generators of G, then it is easy to see that $\Gamma_i G$ is the normal subgroup of G generated by the i-fold commutators of elements of S. Thus Γ is finitely generated as a normal subgroup of G, and the lemma follows at once.

PROOF OF THEOREM 4. In view of Lemma 1, it suffices to show that $\alpha: M \to \hat{M}$ is an $H\mathbf{Z}$-map. Let Γ be as in Lemmas 2 and 3 and let $\nu = G/\Gamma$. Note that ν is nilpotent. Let N be the ν-module $H_0(\Gamma, M) = M/I_\Gamma M$, where I_Γ is the augmentation ideal of $\mathbf{Z}[\Gamma]$. It follows easily from Lemma 2 that $\hat{M}$ is a ν-module and that, moreover, $\hat{M}$ can be identified with $\hat{N}$. We therefore obtain a commutative diagram

$$\begin{array}{ccc} H_i(G,M) & \xrightarrow{\alpha_i} & H_i(G,\hat{M}) = H_i(G,\hat{N}) \\ \phi_i \Big\downarrow & & \Big\downarrow \psi_i \\ H_i(\nu,N) & \xrightarrow{\alpha'_i} & H_i(\nu,\hat{N}), \end{array}$$

where ϕ_i and ψ_i are induced by the projections $G \to \nu$ and $M \to N$ and α_i (resp. α'_i) is induced by the canonical map $\alpha: M \to \hat{M}$ (resp. α': $N \to \hat{N}$). Since α'_i is an isomorphism by Theorem 3, the proof will be complete if we show that ϕ_0, ψ_0, and ψ_1 are isomorphisms and that ϕ_1 is an epimorphism.

Now it is trivial to verify that ϕ_0 and ψ_0 are isomorphisms, and ϕ_1 is easily seen to be an epimorphism by means of the Lyndon-Hochschild-Serre spectral sequence

$$E^2_{pq} = H_p(\nu, H_q(\Gamma, M)) \Rightarrow H_{p+q}(G, M).$$

Finally, to see that ψ_1 is an isomorphism, we again use the spectral sequence, but with coefficient module $\hat{N}$, and we obtain an exact sequence

$$H_0(\nu, \Gamma_{ab}\otimes_{\mathbf{Z}}\hat{N}) \to H_1(G, \hat{N}) \xrightarrow{\psi_1} H_1(\nu, \hat{N}) \to 0.$$

But $H_0(\nu, \Gamma_{ab}\otimes\hat{N}) = \Gamma_{ab}\otimes_\nu\hat{N}$ (this follows at once from the definition of tensor product), so it suffices to show that $\Gamma_{ab}\otimes_\nu\hat{A} = 0$ for any finitely generated ν-module A. Now $A \mapsto \Gamma_{ab}\otimes_\nu\hat{A}$ is right exact (2.1, Prop. 3(i)), so we may assume $A = R = \mathbf{Z}[\nu]$. But then $\Gamma_{ab}\otimes_\nu\hat{A}$ is simply the completion of the right R-module Γ_{ab} (Prop. 3_r(ii) and Lemma 3); since $(G, \Gamma) = \Gamma$, we conclude that $\Gamma_{ab}I = \Gamma_{ab}$, so $\hat{\Gamma}_{ab} = 0$, as required. [An alternative proof that $\Gamma_{ab}\otimes_\nu\hat{A} = 0$, which does not depend on the Artin-Rees property or on the finiteness of A, can be based on the short exact sequence given in the remark at end of Section 2.1.]

3. Applications to topology

The applications we will give concern the structure of *homology equivalences*, i.e., of maps which induce isomorphisms on integral homology.

3.1. *Prenilpotent spaces*

Recall that a CW-complex X is said to be *nilpotent* if X is connected, π_1X is nilpotent, and π_nX is a nilpotent π_1X-module for $n > 1$. (Thus π_nX for $n > 1$ is annihilated by some power of the augmentation ideal of $\mathbf{Z}[\pi_1X]$, cf. Section 1.) A CW-complex X is called *prenilpotent* if there is a homology equivalence $f: X \to Y$ with Y nilpotent, or, equivalently, if the $H_*(-, \mathbf{Z})$-localization of X in the sense of Bousfield [2] is nilpotent. Prenilpotent spaces are studied in [7], where it is shown that, for CW-complexes of finite type (i.e. with finitely many cells in each dimension), one can give an intrinsic characterization of prenilpotence. See also [9], where some examples of prenilpotent spaces are discussed.

In case Y is the circle S^1, X is called a *homology circle*. The analysis of homology circles [6] depends heavily on the fact that (for trivial reasons) the Serre spectral sequence of $f: X \to S^1$ collapses, i.e. $H_p(S^1, H_q(F)) = 0$ for $q > 0$, where F is the homotopy fibre of f. (Note: We are dealing here with homology with *local coefficients*.) The purpose of this section is to prove an analogous collapsing theorem in the general case:

THEOREM 5. *Let X be a prenilpotent space of finite type and let F be the homotopy fibre of a homology equivalence $f: X \to Y$ with Y nilpotent. Then for all $q > 0$ and all p,*

$$E^2_{pq} = H_p(Y, H_q(F)) = 0.$$

The proof will use the following topological analogue of Theorem 3 (Section 2.2):

PROPOSITION 4. *Let Y be a nilpotent space with $\pi_1 Y$ finitely generated. If M is any π_1 Y-module, then $H_*(Y, M \xrightarrow{\approx} H_*(Y, \bar{M})$; if M is finitely generated then $H_*(Y, M) \xrightarrow{\approx} H_*(Y, \hat{M})$.*

(See Section 2.2 for the definitions of $\bar{M}$ and $\hat{M}$.)

The second assertion of proposition follows from the first, in view of 2.1, Prop. 3(ii). To prove the first assertion we use the refined Postnikov tower of Y ([3], Chap. II, §4, Prop. 4.7):

$$\cdots \to Y_i \xrightarrow{p_i} Y_{i-1} \to \cdots \to Y_1 = K(\nu, 1),$$

where $\nu = \pi_1 Y$. Here p_i is a *principal fibration* with fibre F_i of the form $K(A_i, n_i)$ where $2 \leqq n_i \nearrow \infty$, and $Y = \varprojlim Y_i$. It suffices to prove by induction on i that $H_*(Y_i, M) \xrightarrow{\approx} H_*(Y_i, \bar{M})$ for any ν-module M. The case $i = 1$ being true by Theorem 3, we may assume that $i > 1$ and that the result is known for Y_{i-1}. Consider the map of Serre spectral sequences (with local coefficients) induced by the coefficient homomorphism $M \to \bar{M}$:

$$\begin{array}{ccc} H_p(Y_{i-1}, H_q(F_i, M)) & \Rightarrow & H_{p+q}(Y_i, M) \\ \downarrow & & \downarrow \\ H_p(Y_{i-1}, H_q(F_i, \bar{M})) & \Rightarrow & H_{p+q}(Y_i, \bar{M}). \end{array}$$

Note that the groups $H_q(F_i, —)$ which occur here are ordinary homology groups with constant coefficients [F_i is simply connected]; note further that the action of $\nu = \pi_1 Y_{i-1}$ on $H_q(F_i, —)$ comes entirely from the action of ν on the coefficient module. [The action of $\pi_1 Y_{i-1}$ on F_i (in the homotopy category) is trivial, since p_i is a principal fibration with connected fibre.] Therefore, in view of the flatness of $\hat{R}$ over $R = \mathbf{Z}[\nu]$ (2.1, Prop. 3(iii)), we have isomorphisms of ν-modules

$$H_q(F_i, \bar{M}) = H_q(F_i, \hat{R} \otimes_R M) \approx \hat{R} \otimes_R H_q(F_i, M) = \overline{H_q(F_i, M)}.$$

The induction hypothesis now implies that the above map of spectral sequences is an isomorphism on E^2, hence on E^∞, which completes the proof.

We will also need the following lemma:

LEMMA. *Let G be a group such that $\mathbf{Z}[G]$ is noetherian and let M and N be finitely generated G-modules, one of which is finitely generated over $\mathbf{Z}$. Then $M \otimes N$ and $M * N$ are finitely generated G-modules.*

(Here $M\otimes N$ and $M * N$ are the tensor and torsion products over $\mathbf{Z}$, with the usual (diagonal) action of G.)

Assume, for example, that M is finitely generated over $\mathbf{Z}$, and let $(F_i)_{i\geq 0}$ be a free resolution of N over $\mathbf{Z}[G]$, with each F_i finitely generated. Then $M\otimes N$ and $M * N$ can be computed as homology groups of the complex $M\otimes_{\mathbf{Z}}F$; the lemma therefore follows from the easily verified fact that $M\otimes_{\mathbf{Z}}\mathbf{Z}[G]$ is finitely generated over G.

PROOF OF THEOREM 5. We begin with two preliminary observations:

(a) *$\pi_1 f:\pi_1 X\to\pi_1 Y=\nu$ is surjective; hence, in particular, ν is finitely generated and F is connected.* This follows from the surjectivity of $H_1 f: H_1 X\to H_1 Y$, by an argument analogous to that of Lemma 1 (a) of the appendix. [In fact, one actually knows ([5], Prop. 5.1 and first paragraph of Section 6) that $\pi_1 X$ is prenilpotent and that $\nu\approx\pi_1 X/\Gamma$, in the notation of Section 2.3.]

(b) *$H_n(F)$ is a finitely generated ν-module for each n.* In fact, let $p;\tilde{Y}\to Y$ be the universal cover of Y, and consider the pull back $\tilde{f}$ of f to a map over $\tilde{Y}$:

$$\begin{array}{ccc} \tilde{X} & \to & X \\ \tilde{f}\downarrow & & \downarrow f \\ \tilde{Y} & \underset{p}{\to} & Y. \end{array}$$

(thus $\tilde{X}$ is a regular covering space of X, with covering group ν.) Then F is also the homotopy fibre of $\tilde{f}$. Moreover, since ν acts as a group of automorphisms of the map $\tilde{f}$, the Serre spectral sequence

$$E^2_{pq}=H_p(\tilde{Y},H_q(F))\Rightarrow H_{p+q}(\tilde{X})$$

is a spectral sequence of ν-modules. Now $H_n(\tilde{Y})$ is finitely generated over $\mathbf{Z}$ for all n by Cor. 1 of Prop. 5 of the appendix; and $H_n(\tilde{X})$is finitely generated over ν for all n, since the cellular chain complex of $\tilde{X}$ is a complex of finitely generated modules over the noetherian ring $\mathbf{Z}(\nu]$.

Assertion (b) therefore follows at once by a standard mod $\mathscr{C}$ spectral sequence argument (cf. [18]), where $\mathscr{C}$ is the class of finitely generated ν-modules. [Note: the crucial point here is that if N is a finitely generated ν-module, then $H_p(\tilde{Y},N)$, which is a ν-module via the action of ν on $\tilde{Y}$ and on N, is finitely generated. This follows from the universal coefficient theorem and the above lemma.]

We can now prove Theorem 5 by induction on q. Thus assume that $E^2_{pq'}=H_p(Y,H_{q'}(F))=0$ for $0<q'<q$ and all p. Then the edge isomorphism

$H_*(X)\xrightarrow{\approx} H_*(Y)$ implies that $E^2_{0q}=0$. (We are using here the fact that, by (a), F is connected.) But $E^2_{0q}=H_0(\nu, H_q(F))$, so $H_q(F)^\wedge=0$, and hence (by (b) and Prop. 4) $H_p(Y, H_q(F))=0$ for all p, as required.

3.2. *Homology circles and knot complements*

Let X be a homology circle of finite type, and assume that $\pi=\pi_1 X\approx \mathbf{Z}$. Let $\alpha=\pi_n X$ $(n\geqq 2)$ be the first non-zero higher homotopy group of X. Then α is a finitely generated π-module and is *perfect*, i.e., $H_0(\pi,\alpha)=0$; moreover, these are the only conditions on α, since Kervaire [13] has shown that any finitely generated perfect π-module can arise in this way from a homology circle. Our purpose in this section is to give a similar analysis of the module $\beta=\pi_{n+1}X$. Under suitable finiteness assumptions we will show that the I-adic completion $\hat\beta$ is determined by α and that there are no further conditions on β (see below for a precise statement). In particular, the ' perfect part" $I^\infty\beta$ (cf. 2.1, Prop. 3 (iv)) can be arbitrary.

We remark that the results of this section can easily be translated into results about (higher) knot complements. In fact, it is well-known that every knot complement is a homology circle. Conversely, if X is a homology circle which is a finite complex of dimension r, and if $\pi_1 X$ is generated by the conjugates of a single element (e.g., if $\pi_1 X=\mathbf{Z}$), then X is $(m-r)$-equivalent to the complement of an $(m-2)$-sphere in S^m for sufficiently large m ([20], p. 17, Th. 1.7).

THEOREM 6. *Assume that $H_{n+2}(\alpha,n)$ is a finitely generated π-module. Then $\beta=\pi_{n+1}X$ is a finitely generated π-module and $\hat\beta\approx H_{n+2}(\alpha,n)^\wedge$. Moreover, if $\phi:\beta\to\beta'$ is a π-module homorphism with β' finitely generated and $\hat\phi;\hat\beta\to\hat\beta'$ an isomorphism, then one can attach finitely many cells to X to obtain a homology circle X' such that f_*: $\pi_i X\to\pi_i X'$ is an isomorphism for $i\leqq n$ and is equivalent to ϕ for $i=n+1$, where $f;X\to X'$ is the inclusion.*

REMARK. If $n\geqq 3$ then the hypothesis on α holds automatically and the conclusion concerning β simply says that β is perfect. In fact, if $n\geqq 3$ then $H_{n+2}(\alpha,n)\approx\alpha/2\alpha$, which is finitely generated and perfect. [More generally, one can show that $H_{n+k}(\alpha,n)$ is finitely generated and perfect for $n>k$.]

PROOF OF THEOREM 6. Let $\tilde X$ be the universal cover of X and let $p:\tilde X\to K(\alpha,n)$ be the canonical map of $\tilde X$ to the first non-trivial space in its Postnikov decomposition. From the Serre spectral sequence of p we obtain an exact sequence of π-modules,

$$H_{n+2}(\tilde X)\to H_{n+2}(\alpha,n)\to\beta\to H_{n+1}(\tilde X).$$

Since X is a complex of finite type, $H_{n+1}(\tilde{H})$ is a finitely generated π-module; the finite generation of β therefore follows from that of $H_{n+2}(\alpha, n)$. Furthermore, it is easy to see that $H_q(\tilde{X})$ is perfect for $q > 0$, so $H_{n+1}(\tilde{X})^\wedge = 0 = H_{n+2}(\tilde{X})^\wedge$. We therefore obtain an isomorphism $H_{n+2}(\alpha, n)^\wedge \xrightarrow{\approx} \hat{\beta}$ by applying the completion functor to the above exact sequence (2.1, Prop 3(i)). Finally, let $\phi : \beta \to \beta'$ be as in the statement of the theorem; then ϕ is an $H\mathbf{Z}$-map of π-modules (2.2, Cor. 1 of Th. 3), so Lemma 6.2 of [2] implies that we can attach cells to X to obtain a space X' with the desired properties. It is clear from the proof of that lemma that only finitely many cells are required.)

Appendix. The homology (mod $\mathscr{C}$) **of a regular covering space**

In this appendix we will prove a result about covering spaces (Prop. 5 below), the first corollary of which was needed in Section 3.1, and we will show it can be used to extend to nilpotent spaces Serre's mod $\mathscr{C}$ Hurewicz theorem for simply connected spaces (see Prop. 6 below).

Let R be a commutative ring and let $\mathscr{C}$ be a Serre's class of R-modules, i.e., $\mathscr{C}$ contains 0 and is closed under submodules, quotient modules, and extensions. Assume that $\mathscr{C}$ has the following property:

(*) *If $M, N \in \mathscr{C}$, then $\mathrm{Tor}_p^R(M, N) \in \mathscr{C}$ for all $p \geqq 0$.*

PROPOSITION 5. *Let $: \tilde{X} \to X$ be a regular covering map of path-connected spaces, let G be the group of covering transformations, and let n be a positive integer. Assume that $H_p(G, R) \in \mathscr{C}$ for each $p > 0$ and that $H_i(\tilde{X}, R)$ is a nilpotent G-module for $i < n$. Then the following conditions are equivalent:*

(i) *$H_i(\tilde{X}, R) \in \mathscr{C}$ for $1 \leqq i < n$.*

(ii) *$H_i(X, R) \in \mathscr{C}$ for $1 \leqq 1 < n$.*

Furthermore, (i) *and* (ii) *imply:*

(iii) *p induces a $\mathscr{C}$-isomorphism*

$$H_n(\tilde{X}, R)_G \underset{\mathscr{C}}{\xrightarrow{\approx}} H_n(X, R).$$

(Note: For any $R[G]$-module M, we denote by M_G the R-module $H_0(G, M) = M/IM$, where I is the augmentation ideal of $R[G]$.)

COROLLARY 1. *Let X be a nilpotent space and let $\tilde{X}$ be its universal cover. If $H_i(X)$ is finitely generated for each i then so is $H_i(\tilde{X})$.*

In fact, it is not hard to show that $H_i(\tilde{X})$ is a nilpotent G-module each i, where $G = \pi_1 X$. [Prove inductively that G acts nilpotently on the homology of the Postnikov approximations to $\tilde{X}$. Alternatively, apply Lemma 5.4 of [3], Chap. II, §5, to the fibration $\tilde{X} \to X \to K(G,1)$.] Moreover, since $H_1(G) = H_1(X)$ is finitely generated, Lemma 1(a) below shows that $\Gamma_i G/\Gamma_{i+1} G$ is finitely generated for each i, and it follows easily that $H_p(G)$ is finitely generated for each p. We can therefore apply the proposition to the covering map $\tilde{X} \to X$, with $R = \mathbf{Z}$ and $\mathscr{C}$ equal to the class of finitely generated abelian groups, and the corollary follows at once.

Corollary 2. *Let $f: \tilde{X} \to X$ be a regular covering map of degree a power of a prime p. If $H_i(X; \mathbf{Z}/p\mathbf{Z})$ is finite for each i, then so is $H_i(\tilde{X}; \mathbf{Z}/p\mathbf{Z})$.*

In fact, one knows that if G is a finite p-group and k is a field of characteristic p, then every $k[G]$-module is nilpotent (cf. [19], Chap. IX, §1, Cor. of Th. 2). The corollary therefore follows from the proposition, applied with $R = \mathbf{Z}/p\mathbf{Z}$ and $\mathscr{C}$ equal to the class of finite R-modules.

The proof of Proposition 5 requires two lemmas.

Lemma 1. *Let G be a group such that $H_1(G, R) \in \mathscr{C}$.*

(a) *Letting $\{\Gamma_i G\}_{i \geq 1}$ be the lower central series of G, the R-modules $R \otimes_{\mathbf{Z}} (\Gamma_i G/\Gamma_{i+1} G)$ are in $\mathscr{C}$ for all $i \geqq 1$.*

(b) *If M is an $R[G]$-module such that $M_G \in \mathscr{C}$, then $I^i M/I^{i+1} M \in \mathscr{C}$ for all $i \geqq 0$, where I is the augmentation ideal of $R[G]$. In particular, if M is nilpotent, then $M \in \mathscr{C}$.*

To prove (a), recall that the commutator map $G \times \Gamma_i G \to \Gamma_{i+1} G$ induces by passage to the quotient a $\mathbf{Z}$-bilinear map $G_{ab} \times \Gamma_i G/\Gamma_{i+1} G \to \Gamma_{i+1} G/\Gamma_{i+2} G$, where G_{ab} is the abelianization $\Gamma_1 G/\Gamma_2 G$ of G, cf. [12], p. 329, for example. This yields by extension of scalars an R-module homomorphism

$$(R \otimes_{\mathbf{Z}} G_{ab}) \otimes_R (R \otimes_{\mathbf{Z}} (\Gamma_i G/\Gamma_{i+1} G)) \to R \otimes_{\mathbf{Z}} (\Gamma_{i+1} G/\Gamma_{i+2} G),$$

which is clearly surjective by the definition of $\Gamma_{i+1} G$. Since $R \otimes_{\mathbf{Z}} G_{ab} = H_1(G, R) \in \mathscr{C}$, the result follows by induction on i from the fact that $\mathscr{C}$ is closed under tensor products and quotients.

Similarly, to prove (b), note that the multiplication map $I \otimes_R I^i M \to I^{i+1} M$ induces an epimorphism

$$(I/I^2)\otimes_R(I^iM/I^{i+1}M)\to I^{i+1}M/I^{i+2}M.$$

Since $I/I^2\approx H_1(G,R)$ (cf. [4], p. 184, formula (4)), (b) follows at once by induction on i.

LEMMA 2. *Let G be a group such that $H_p(G,R)\in\mathscr{C}$ for all $p>0$. If M is a nilpotent $R[G]$-module whose underlying R-module is in $\mathscr{C}$, then $H_p(G,M)\in\mathscr{C}$ for all $p\geqq 0$.*

Since G acts trivially on the quotients $I^iM/I^{i+1}M$, which are in $\mathscr{C}$, it suffices to consider the case where G acts trivially on M. In this case the result follows from (*) and the universal coefficient spectral sequence

$$E^2_{pq}=\operatorname{Tor}^R_p(H_q(G,R),M)\Rightarrow H_{p+q}(G,M)$$

([11], Chap. I, Th. 5.5.1).

PROOF OF PROPOSITION 5. We will use the spectral sequence

$$E^2_{pq}=H_p(G,H_q(\tilde{X},R))\Rightarrow H_{p+q}(X,R).$$

Note that the hypotheses imply that $E^2_{p0}\in\mathscr{C}$ for $p>0$. Assuming now that (i) holds, Lemma 2 implies that $E^2_{pq}\in\mathscr{C}$ for $1\leqq q<n$ and all $p\geqq 0$, whence, by a standard spectral sequence argument, (ii) and (iii) hold. It remains to prove that (ii) implies (i). Assuming that (ii) holds, and assuming inductively that $H_j(\tilde{X},R)\in\mathscr{C}$ for $1\leqq j<i$ (where i is fixed, $i\leqq j<n$), it follows from what we have just proved that is there a $\mathscr{C}$-isomorphism

$$H_i(\tilde{X},R)_G\underset{\mathscr{C}}{\overset{\approx}{\to}}H_i(X,R),$$

hence $H_i(\tilde{X},R)_G\in\mathscr{C}$. But $H_i(\tilde{X},R)$ is a nilpotent $R[G]$-module, so Lemma 1 (b) implies that $H_i(\tilde{X},R)\in\mathscr{C}$, as required.

We now specialize to the case $R=\mathbf{Z}$, and we assume that $\mathscr{C}$ satisfies, in addition to (*), the following property:

(**) *If $G\in\mathscr{C}$ then $H_p(G)\in\mathscr{C}$ for all $p>0$.*

If G is a nilpotent group such that $\Gamma_iG/\Gamma_{i+1}G\in\mathscr{C}$ for each $i\geqq 1$, then we will say, by abuse of language, that $G\in\mathscr{C}$. It is easy to see that the conclusion of (**) continues to hold for such a G.

If X is a path-connected space, then $\pi_1(X, x_0)$ operates on $\pi_n(X, x_0)$ for $n \geqq 2$ and we set

$$\pi'_n X = \pi(X, x_0)_{\pi_1(X, x_0)}.$$

(Here x_0 is an arbitrary basepoint, but the right-hand side is independent of x_0, up to canonical isomorphism.)

PROPOSITION 6. *Let X be a nilpotent space. For any integer $n \geqq 2$, the following conditions are equivalent*:

(i) $\pi_i X \in \mathscr{C}$ *for* $1 \leqq i < n$.

(ii) $H_i X \in \mathscr{C}$ *for* $1 \leqq i < n$.

Furthermore, (i) *and* (ii) *imply*:

(iii) *The Hurewicz map* $\pi'_n X \to H_n X$ *is a* $\mathscr{C}$*-isomorphism.*

Assume first that (ii) holds. Then $(\pi_1 X)_{ab} \in \mathscr{C}$, so $\pi_1 X \in \mathscr{C}$ by Lemma 1 (a). Letting $p : \tilde{X} \to X$ be the universal cover of X, it follows that the hypotheses of Proposition 5 are satisfied (cf. proof of Cor. 1 above). We conclude that $H_i \tilde{X} \in \mathscr{C}$ for $1 \leqq i < n$ and that p induces a $\mathscr{C}$-isomorphism

$$(H_n \tilde{X})_G \underset{\mathscr{C}}{\overset{\approx}{\to}} H_n X, \tag{1}$$

where $G = \pi_1 X$. The mod $\mathscr{C}$ Hurewicz theorem for simply connected spaces [18] now implies that $\pi_i \tilde{X} \in \mathscr{C}$ for $2 \leqq i < n$ and that the Hurewicz map is a $\mathscr{C}$-isomorphism

$$\pi_n \tilde{X} \underset{\mathscr{C}}{\overset{\approx}{\to}} H_n \tilde{X}. \tag{2}$$

Since $\pi_i \tilde{X} \overset{\approx}{\to} \pi_i X$ for $i \geqq 2$, (i) follows at once and (iii) follows from (1) and (2) together with the easily verified fact that a $\mathscr{C}$-isomorphism $M \to N$ of nilpotent G-modules induces a $\mathscr{C}$-isomorphism $M_G \to N_G$. [This can be deduced from Lemma 2 above.] Thus (ii) implies (i) and (iii). The implication (i) $\Rightarrow$ (ii) is proved similarly.

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